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About the Recognition and Reconstruction Two Unknown Functions From Known Their Tandem

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***Abstract*— Paper is devoted to the area of processing of physical signals and data or artificial perception. The incoming data are presented in the form of defined sum of two unknown functions, spaced on finite carrier. Practical examples of the functions searched for can be presented: (a) by very closely spaced *two* atomic spectrums of radiation; (b) by the output signal of some telescope, scanning *two* point-like sources with angle distance between each other much less, than the angle width of main petal of telescope's directivity pattern. So the presented result can be classified as some case of the known problem of superresolution. It is shown, that *two* functions of tandem (mutually overlapping each other) can be reconstructed *separately* by using only *three* it is correct chosen numbers. The class of functions (and their tandems), which are overlapping each other and allows single reconstruction of both searched functions and also conditions of the correctness of the decision are formulated. The method of separation of two functions had been tested numerically on the variety of typical practical examples. The stability of suggested numerical procedure is confirmed in the presence of uncorrelated errors of measurements and calculations at each point of tandem.**

***Index Terms*—About four key words or phrases in alphabetical order, separated by commas.**

I. Introduction

Usually the problem reconstruction of signals is considered in the presence of noise and interpreted as extraction of signal $U(z)$ covered by noise, so the measured signal can be of the form $W(z) = U(z) + \xi(z)$, where $\xi(z)$ - stationary noise with mean-square value

$$\sigma_\xi = <\xi^2(z)>^{1/2}$$

correlation interval

$$\tau_\xi << L \text{ [1].}$$

Below we will consider some untraditional statement of the problem.

II. Statement of the Problem

Usually the problem reconstruction of signals is considered in the presence of noise and interpreted as extraction of signal Statement of the problem

We have only two unknown smooth and mutually overlapping functions $U(z)$, $V(z)$, which are forming exactly predetermined smooth source function $W(z) = U(z) + V(z)$ on the carrier $|z| < L$. We find the single function $U(z)$ and single function $V(z)$ in Figure 1-a .

Then we do important assumption: $W(z)$ is presented by the tandem (all variables and numbers, used below, are assumed dimensionless) of two smooth functions $U(z)$ and $V(z)$, which are overlapping each other at any point of the same carrier $|z| \le L$. No any point $z = z^*$ on above carrier where function $U(z^*) = 0$ at $V(z^*) \ne 0$ or $V(z^*) = 0$ at $U(z^*) \ne 0$. Itself determination $W(z)$ as "two functions" nothing do not signify. In realities possible to name such combination "one function", or suppose, that $W(z)$ consists of tens of functions. To avoid the uncertainty, we must say the important thing: function $U(z)$ has a centre of symmetry z_U, and function $V(z)$ has a centre of symmetry z_V. In addition the searched functions U, V are assumed mutually linearly independent, and we must find *single solution* for tandem U, V on chosen class of functions (see below in Section IV). Such problems can appear in investigations of closely spaced spectrums of atomic radiation or in resolutions of astronomic sources by radio antennas [2]-[4]. Below we suppose negligible weak noise (see Figure 1-b)

$$\sigma_\xi << A_W, A_U, A_V, \tag{1}$$

where symbols A_W >0, A_U >0, A_V >0 mean the characteristic magnitudes

$$A_W = [\textstyle\int_{-L}^{L} W^2(z)dz/(2L)]^{1/2},$$

$$A_U = [\textstyle\int_{-L}^{L} U^2(z)dz/(2L)]^{1/2},$$

$$A_V = [\textstyle\int_{-L}^{L} V^2(z)dz/(2L)]^{1/2}$$

of values $W(z)$, $U(z)$, $V(z)$, following (1),

$$W(z) = U(z) + V(z). \tag{2}$$

Both component $U(z)$, $V(z)$ of tandem is assumed function, symmetrical relating to some point (symmetry axis) z_U, z_V correspondingly. Figure 1-c illustrates the cases of generalized symmetry of the following form: value of function $U(-z)$ (or $V(-z)$) are predetermined by the value of function $U(z)$ (or $V(z)$) by coefficient of symmetry (coefficient of reflection) as was described in the assumption $\hat{\mathbf{H}}$ in the section III).

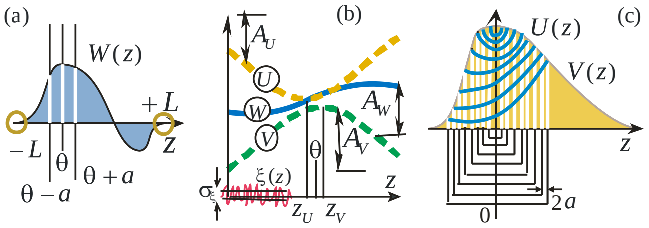


Figure 1

III. Assumptions (hypothesizes)

$\hat{\mathbf{A}}$. Distance $2a$ between centers z_U and z_V of symmetry of functions $U(z)$ and $V(z)$ correspondingly is assumed $a = a_m$ $\theta = \theta_k$ ($\hat{\mathbf{A}} = \mathbf{1}$ if yes, $\hat{\mathbf{A}} = 0$ if no).

$\hat{\mathbf{B}}$. Center point $\theta = (z_V - z_U)/2$ between two centers of symmetry is assumed placed in the point $\theta = \theta_k$ ($\hat{\mathbf{B}} = \mathbf{1}$ if yes, $\hat{\mathbf{B}} = 0$ if no).

$\hat{\mathbf{C}}$. Symmetry center of function $U(z)$ is spaced in the point $z = \theta_k + a_m$ (we assume this hypothesis is true $\hat{\mathbf{C}} = \mathbf{1}$ if yes, $\hat{\mathbf{C}} = 0$ if no).

$\hat{\mathbf{D}}$. Symmetry center of function $V(z)$ is spaced in the point $z = \theta_k - a_m$ (we assume this hypothesis is true $\hat{\mathbf{D}} = \mathbf{1}$ if yes, $\hat{\mathbf{D}} = 0$ if no).

$\hat{\mathbf{E}}$. Points z_U and z_V are spaced sufficiently far from borders $\pm L$. In other words, we assume relation $|\theta \pm a| << L$.

$\hat{\mathbf{F}}$. the value of function $U(z)$ in the start-point $z = \theta$ $U(\theta) = \mu$ (we will check this hypothesis step by step $\hat{\mathbf{F}} = \mathbf{1}$ if yes, $\hat{\mathbf{F}} = 0$ if no).

$\hat{\mathbf{G}}$. Function $\tilde{\Gamma}_U(z) = \Gamma_U(z)$ of symmetry of function $U(z)$ (and $\tilde{U}(z)$) ($\hat{\mathbf{G}} = \mathbf{1}$ if yes, $\hat{\mathbf{G}} = 0$, if no).

$\hat{\mathbf{H}}$. Function $\tilde{\Gamma}_V(z) = \Gamma_V(z)$ of symmetry of function $V(z)$ (and $\tilde{V}(z)$) ($\hat{\mathbf{H}} = \mathbf{1}$, if yes, $\hat{\mathbf{H}} = 0$, if no). Simplest cases of symmetry present $\Gamma_{V,U} = \pm 1$, however one can use more general form of symmetry $U(-z) = \Gamma_U(z)U(z)$, $V(-z) = \Gamma_V(z)V(z)$ (Figure 1-c). Functions $\Gamma_U(z)$, $\Gamma_V(z)$ (coefficients of symmetry, coefficient of reflection,…) must provide one-to-one correspondence of functions $U(z)$ and $U(-z)$, $V(z)$ and $V(-z)$, not breaking their smoothness.

$$\hat{\mathbf{J}}.\ |\tilde{V}(\pm L)| << |W(z)|_{\max},\ |\dot{\tilde{V}}(\pm L)| << |\dot{W}(z)|_{\max}, \quad (3)$$

$$|\tilde{U}(\pm L)| << |W(z)|_{\max},\ |\dot{\tilde{U}}(\pm L)| << |\dot{W}(z)|_{\max} \text{ at } \forall |z| \le L.$$

In other words, functions are assumed smoothly going closely to zero together with their smooth first derivatives (see Figure 1-a).

$\hat{\mathbf{K}}$. Functions $W(z), U(z), V(z)$ are assumed smooth at $\forall |z| \le L$. Their characteristic spatial scales are of the same order with $L/2$.

So at $\hat{\mathbf{A}} \cap \hat{\mathbf{B}} \cap \hat{\mathbf{C}} \cap \hat{\mathbf{D}} \cap \hat{\mathbf{E}} \cap \hat{\mathbf{F}} \cap \hat{\mathbf{G}} \cap \hat{\mathbf{H}} \cap \hat{\mathbf{J}} \cap \hat{\mathbf{K}} = \mathbf{1}$, we will get result $|V(z) - \tilde{V}(z)| << |W(z)|_{\max}$, $|U(z) - \tilde{U}(z)| << |W(z)|_{\max}$, where $\tilde{U}(z)$ and $\tilde{V}(z)$ represents the searched solutions which must be maximally closely to $U(z)$ and $V(z)$ correspondingly.

IV. Tandem Restoration Procedure

Now we consider the special computing procedure, which represents the base of all further investigations. Figure 2-a represents smooth function spiral process of tandem restoration. For the schematic model of pervious concrete problem we try to find a single solution for both functions of tandem.

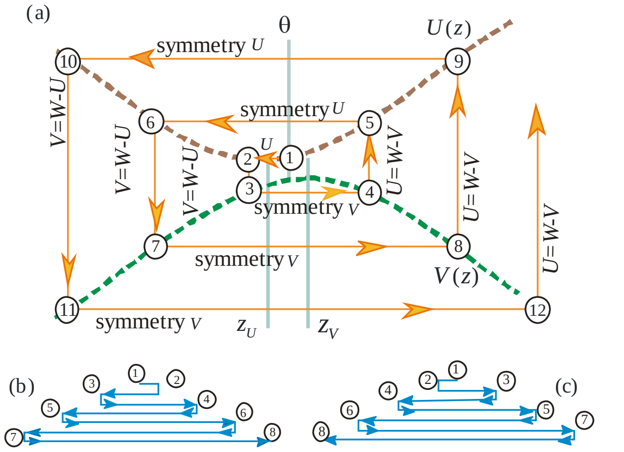


Figure 2

The appearance sequence of interpolation nodes (see below) resembles the linear structure of image light sources, caused by one light point-like source spaced in the point $z = \theta$ between parallel plane mirrors with coordinates $z = \theta \pm a$. Spiral begins (see Fig 2-a) in the point θ and goes to the left. Vertical transitions of spiral line are caused by equation (2), horizontal transitions of spiral line are caused by concrete type of symmetry $\Gamma_U(z)$, $\Gamma_V(z)$ of searched functions $U(z), V(z)$ correspondingly. Table 1 shows the generation of spiral; sequences of interpolation nodes $\vec{x} = \{x_i\}$, $\vec{p} = \{p_i\}$, $\vec{q} = \{q_i\}$ ($i = 1,2,3,\ldots,13$), when starting from point $z = \theta$ to the left (see Figure 2-a,b).

Table I Spiral sequence of interpolation

$x_1 = \theta$	$p_1 = \tilde{\mu}$	$q_1 = W(x_1) - p_1$
$x_2 = \theta - 2\tilde{a}$	$p_2 = p_1 \tilde{\Gamma}_U(\tilde{a})$	$q_2 = W(x_2) - p_2$
$x_3 = \theta + 4\tilde{a}$	$q_3 = q_2 / \tilde{\Gamma}_V(3\tilde{a})$	$p_3 = W(x_3) - q_3$
$x_4 = \theta - 6\tilde{a}$	$p_4 = p_3 \tilde{\Gamma}_U(5\tilde{a})$	$q_4 = W(x_4) - p_4$
$x_5 = \theta + 8\tilde{a}$	$q_5 = q_4 / \tilde{\Gamma}_V(7\tilde{a})$	$p_5 = W(x_5) - q_5$
$x_6 = \theta - 10\tilde{a}$	$p_6 = p_5 \tilde{\Gamma}_U(9\tilde{a})$	$q_6 = W(x_6) - p_6$

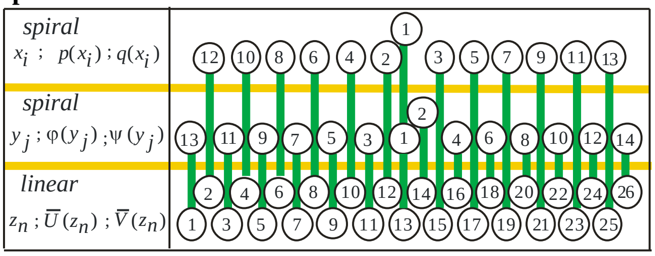
$x_7 = \theta + 12\tilde{a}$	$q_7 = q_6 / \tilde{\Gamma}_V(11\tilde{a})$	$p_7 = W(x_7) - q_7$
$x_8 = \theta - 14\tilde{a}$	$p_8 = q_7 \tilde{\Gamma}_U(13\tilde{a})$	$q_8 = W(x_8) - p_8$
$x_9 = \theta + 16\tilde{a}$	$q_9 = q_8 / \tilde{\Gamma}_V(15\tilde{a})$	$p_9 = W(x_9) - q_9$
$x_{10} = \theta - 18\tilde{a}$	$p_{10} = p_9 \tilde{\Gamma}_U(17\tilde{a})$	$q_{10} = W(x_{10}) - p_{10}$
$x_{11} = \theta + 20\tilde{a}$	$q_{11} = q_{10} / \tilde{\Gamma}_V(19\tilde{a})$	$p_{11} = W(x_{11}) - q_{11}$

Table II shows the generation of spiral sequences $\vec{y} = \{y_j\}$, $\vec{\varphi} = \{\varphi_j\}$, $\vec{\psi} = \{\psi_j\}$ ($j = 1,2,3,...,13$) of interpolation nodes, when starting from point $z = \theta$ to the right (see Figure 2-c).

Table II Spiral sequence of interpolation nodes

$y_1 = \theta$	$\psi_1 = q_1$	$\varphi_1 = p_1$
$y_2 = \theta + 2\tilde{a}$	$\psi_2 = \psi_1 / \tilde{\Gamma}_V(\tilde{a})$	$\varphi_2 = W(y_2) - \psi_2$
$y_3 = \theta - 4\tilde{a}$	$\varphi_3 = \varphi_2 \tilde{\Gamma}_U(3\tilde{a})$	$\psi_3 = W(y_3) - \varphi_3$
$y_4 = \theta + 6\tilde{a}$	$\psi_4 = \psi_3 / \tilde{\Gamma}_V(5\tilde{a})$	$\varphi_4 = W(y_4) - \psi_4$
$y_5 = \theta - 8\tilde{a}$	$\varphi_5 = \varphi_4 \tilde{\Gamma}_U(7\tilde{a})$	$\psi_5 = W(y_5) - \varphi_5$
$y_6 = \theta + 10\tilde{a}$	$\psi_6 = \psi_5 / \tilde{\Gamma}_V(9\tilde{a})$	$\varphi_6 = W(y_6) - \psi_6$
$y_7 = \theta - 12\tilde{a}$	$\varphi_7 = \varphi_6 \tilde{\Gamma}_U(11\tilde{a})$	$\psi_7 = W(y_7) - \varphi_7$
$y_8 = \theta + 14\tilde{a}$	$\psi_8 = \psi_7 / \tilde{\Gamma}_V(13\tilde{a})$	$\varphi_8 = W(y_8) - \psi_8$
$y_9 = \theta - 16\tilde{a}$	$\varphi_9 = \varphi_8 \tilde{\Gamma}_U(15\tilde{a})$	$\psi_9 = W(y_9) - \varphi_9$
$y_{10} = \theta + 18\tilde{a}$	$\psi_{10} = \psi_9 / \tilde{\Gamma}_V(17\tilde{a})$	$\varphi_{10} = W(y_{10}) - \psi_{10}$
$y_{11} = \theta - 20\tilde{a}$	$\varphi_{11} = \varphi_{10} \tilde{\Gamma}_U(19\tilde{a})$	$\psi_{11} = W(y_{11}) - \varphi_{11}$

Table III Renumbering interpolation nodes from spiral structure into linear structure



Using Table III and cubic-spline interpolation [5]-[7], we obtain 26 nodes of interpolation and following continuous functions

$$\tilde{U}(z) = \mathrm{interp}[\mathrm{cspline}(\vec{z},\vec{u}),\vec{z},\vec{u},z], \qquad (4)$$

$$\tilde{V}(z) = \mathrm{interp}[\mathrm{cspline}(\vec{z},\vec{\upsilon}),\vec{z},\vec{\upsilon},z]\ . \qquad (5)$$

Where $\vec{z} = \{z_n\}$, $\vec{U} = \{u_n\}$, $\vec{\upsilon} = \{\upsilon_n\}$ ($n = 1,2,3,...,26$). Spatial period of sequence $\vec{z} = \{z_n\}$ is $2a$. Note, that calculation procedures represented in Tables 1,2,3 guaranteed (at zero noise $\sigma_\xi = 0$) the obvious equations $\bar{U}(z) + \bar{V}(z) = W(z)$ in the nodes $z = z_n$ (at any $\tilde{a}$, $\tilde{\mu}$, $\tilde{\theta}$!), which does not any guarantees for equations $\tilde{U}(z) - U(z) = 0$, $\tilde{V}(z) - V(z) = 0$.

Now our aim is to find a single solution, $V(z)$ for tandem by variation ($i = 1,2,..., N_a$; $j = 1,2,..., N_\mu$; $k = 1,2,..., N_\theta$) and selection of correct combinations of three numbers a_i, μ_j, θ_k. Sorting patterns $\tilde{U}(z) = \tilde{U}(z; a_i; \mu_j; \theta_k)$, $\tilde{V}(z) = \tilde{V}(z; a_i; \mu_j; \theta_k)$, we will search for the tandem $\tilde{U}(z), \tilde{V}(z)$ satisfying all above conditions (assumptions, hypothesizes) $\hat{\mathbf{A}} - \hat{\mathbf{K}}$.

V. Markers of Errors

When searching for the single $\tilde{U}(z)$ and single $\tilde{V}(z)$ we sort all $N_a N_\mu N_\theta >> 1$ patterns (and corresponding all combinations a_i, μ_j, θ_k). Any real combination of key-numbers a_i, μ_j, θ_k includes some small or very small errors. We can't measure immediately errors of tandem's restoration because both functions $U(z), V(z)$ are assumed *unknown priori*.

However, we have found as empirical fact - an errors in key numbers are always accompanied: (1) by the appearance of the fluctuations function on high spatial frequency $\sim \pi / \tilde{a}$ and its harmonics (the breach of the hypothesis $\hat{\mathbf{K}}$ to smoothness of $U(z)$, $V(z)$, see Section III.); (2) by growing of the modules function $|U(\pm L)|$, $|V(\pm L)|$ on edges $\pm L$ of carrier $|z| < L$ (the breach of the hypothesis $\hat{\mathbf{J}}$, see Section III). Thereby, we have built the quantitative criterion $\varepsilon = \varepsilon[\tilde{a}; \tilde{\mu}; \tilde{\theta}]$ of the choice correct key numbers:

$$\varepsilon = \varepsilon[\tilde{a}; \tilde{\mu}; \tilde{\theta}] = \varepsilon_U + \varepsilon_V\ , \qquad (6)$$

$$\varepsilon_U = \left(I_{UC}^2 + I_{US}^2\right)^{1/2} (2L)^{-1} + \left|\tilde{U}(-L)\right| + \left|\tilde{U}(+L)\right|\ , \qquad (7)$$

$$\varepsilon_V = \left(I_{VC}^2 + I_{VS}^2\right)^{1/2} (2L)^{-1} + \left|\tilde{V}(-L)\right| + \left|\tilde{V}(+L)\right|\ , \qquad (8)$$

$$I_{UC} = \int_{-L}^{L} \tilde{U}(z) \cos(z\pi / \tilde{a}) dz\ , \qquad (9)$$

$$I_{VC} = \int_{-L}^{L} \tilde{V}(z) \cos(z\pi / \tilde{a}) dz\ , \qquad (10)$$

$$I_{US} = \int_{-L}^{L} \tilde{U}(z) \sin(z\pi / \tilde{a}) dz\ , \qquad (11)$$

$$I_{VS} = \int_{-L}^{L} \tilde{V}(z) \sin(z\pi / \tilde{a}) dz\ . \qquad (12)$$

Sorting all values a_i, μ_j, θ_k ($i = 1,2,..., N_a$; $j = 1,2,..., N_\mu$; $k = 1,2,..., N_\theta$), we search for the minimum of the calculated value ε. But value (and its minimum) ε not error of restoration of searched tandem $\tilde{U}(z)$, $\tilde{V}(z)$. The minimum of the value $\varepsilon[\tilde{a}; \tilde{\mu}; \tilde{\theta}]$, reached by choosing of *three correct key-numbers* $\tilde{a} = a_i = a$, $\tilde{\mu} = \mu_j = \mu$, $\tilde{\theta} = \theta_k = \theta$, give us the tandem, most closing to the correct true functions

$$\left|U-\tilde{U}\right|,\left|V-\tilde{V}\right|<10^{-5}\left|W\right|_{\max} \qquad (13)$$

(see below).

Sensitivities $\Delta(U,\tilde{a})$, $\Delta(V,\tilde{a})$, $\Delta(U,\tilde{\mu})$, $\Delta(V,\tilde{\mu})$, $\Delta(U,\tilde{\theta})$, $\Delta(V,\tilde{\theta})$ ((14)-(19)) of restored functions $\tilde{U}(z),\tilde{V}(z)$ to the *key-numbers* errors $\bar{a}=\tilde{a}-a$, $\bar{\mu}=\tilde{\mu}-\mu$, $\bar{\theta}=\tilde{\theta}-\theta$ we can use *only after tandem reconstruction* (post factum, using true model functions $U(z)$, $V(z)$ and on the base of *key-numbers* hypothesizes $\tilde{a}$, $\tilde{\mu}$, $\tilde{\theta}$ and sorting of many patterns) given by minimums of errors markers $\varepsilon[\tilde{a};\tilde{\mu};\tilde{\theta}]$).

$$\Delta(U,a)=\underset{|\bar{a}|\to 0}{\mathrm{Lim}}\left[\underset{|z|\le L}{\mathrm{Max}}\left|\tilde{U}-U\right|/\left|\bar{a}\right|\right], \qquad (14)$$

$$\Delta(V,a)=\underset{|\bar{a}|\to 0}{\mathrm{Lim}}\left[\underset{|z|\le L}{\mathrm{Max}}\left|\tilde{V}-V\right|/\left|\bar{a}\right|\right], \qquad (15)$$

$$\Delta(U,\mu)=\underset{|\bar{\mu}|\to 0}{\mathrm{Lim}}\left[\underset{|z|\le L}{\mathrm{Max}}\left|\tilde{U}-U\right|/\left|\bar{\mu}\right|\right], \qquad (16)$$

$$\Delta(V,\mu)=\underset{|\bar{\mu}|\to 0}{\mathrm{Lim}}\left[\underset{|z|\le L}{\mathrm{Max}}\left|\tilde{V}-V\right|/\left|\bar{\mu}\right|\right], \qquad (17)$$

$$\Delta(U,\theta)=\underset{|\bar{\theta}|\to 0}{\mathrm{Lim}}\left[\underset{|z|\le L}{\mathrm{Max}}\left|\tilde{U}-U\right|/\left|\bar{\theta}\right|\right], \qquad (18)$$

$$\Delta(V,\theta)=\underset{|\bar{\theta}|\to 0}{\mathrm{Lim}}\left[\underset{|z|\le L}{\mathrm{Max}}\left|\tilde{V}-V\right|/\left|\bar{\theta}\right|\right]. \qquad (19)$$

Further we will consider several examples of tandem's components with different types of symmetry, with different relations of amplitudes, with different velocity of the decline to edge of the carrier.

VI. Symmetric & Antisymmetric Lorentz's Functions in Tandem

In this section (and next section V) we consider [2], [3] variants of Lorentz's functions. Now consider combination of symmetric and antisymmetrical functions

$$U(z)=1.8/\left[1+0.005(z-2)^2\right] \qquad (20)$$

$$V(z)=2.2\sin[0.1(z+2)]/\left[1+0.005(z+2)^2\right] \qquad (21)$$

with coefficients of symmetry

$$\Gamma_V(z)=-1,\ \Gamma_U(z)=1 \qquad (22)$$

and true *key numbers*

$$a=2,\ \theta=0,\ \mu=U(0). \qquad (21)$$

Errors markers $\varepsilon[\bar{a};\mu;\theta]$, $\varepsilon[a;\bar{\mu};\theta]$, $\varepsilon[a;\mu;\bar{\theta}]$ are pointing to the values $\tilde{a}$, $\tilde{\mu}$, $\tilde{\theta}$ closest with the parameters a, μ, θ of correct restoration. They are represented in Fig. 3.

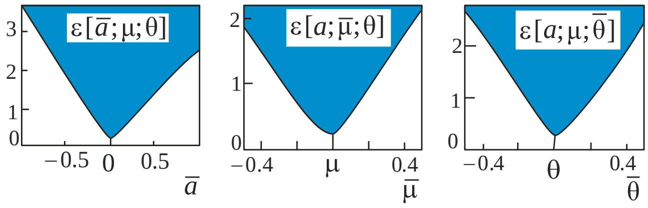


Figure 3

Results $\tilde{U}(z),\tilde{V}(z)$ of restoration of searched functions $U(z),V(z)$ (a) from their tandem $W(z)$ (b), using exact values a,μ,θ, are represented in Fig. 4.

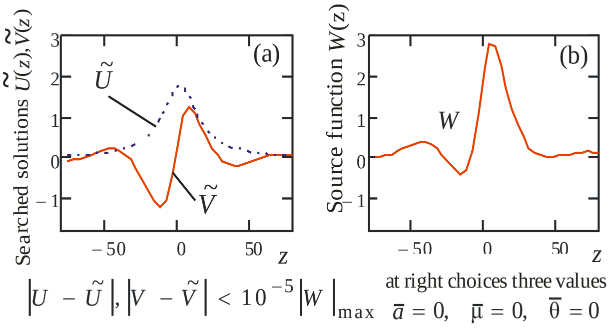


Figure 4

One can note, that with any errors $\tilde{U}(z)-U(z)$, $\tilde{V}(z)-V(z)$ of restoration, the sum of tandem $U(z)-V(z)$ is saved equal $W(z)$, following the calculation procedure described in Section IV (excluding, of course, the interpolation errors).

Results $\tilde{U}(z),\tilde{V}(z)$ of restoration of searched functions $U(z),V(z)$ (a) and errors of restoration $\tilde{U}(z)-U(z)$, $\tilde{V}(z)-V(z)$ (b), admitting only one nonzero error $\bar{a}\neq 0$, ($\Delta(U,a)\approx\Delta(V,a)\approx 0.4$, see (14), (15), are represented in Fig. 6.

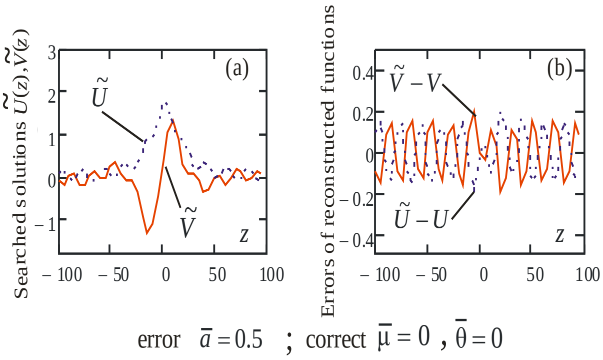
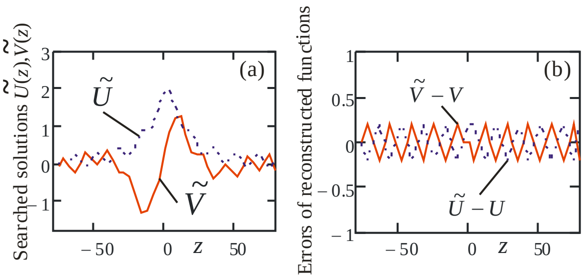


Figure 5

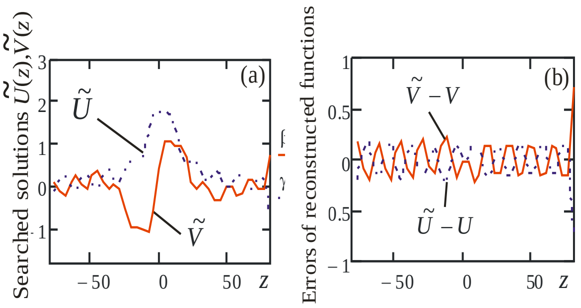
Results $\tilde{U}(z),\tilde{V}(z)$ of restoration of searched functions $U(z),V(z)$ (a) and errors of restoration $\tilde{U}(z)-U(z)$, $\tilde{V}(z)-V(z)$ (b), admitting only one nonzero error $\bar{\mu}\neq 0$ ($\Delta(U,\mu)\approx\Delta(V,\mu)\approx 1.25$, see (16), (17)), are represented in Fig. 6.



error $\bar{\mu}=0.2$; correct $\bar{a}=a$, $\bar{\theta}=0$

Figure 6

Results $\tilde{U}(z), \tilde{V}(z)$ of restoration of searched functions $U(z), V(z)$ (a) and errors of restoration $\tilde{U}(z)-U(z)$, $\tilde{V}(z)-V(z)$ (b), admitting only one nonzero error $\bar{\theta} \neq 0$, ($\Delta(U,\theta) \approx \Delta(V,\theta) \approx 0.5$, see (18), (19)), are represented in Fig. 7.



error $\bar{\theta}=0.4$; correct $\bar{a}=a$, $\bar{\mu}=0$

Figure 7

VII. Antisymmetric & Antisymmetric Lorentz's Functions in Tandem

For the following model functions

$$V(z)=2.2\sin[0.1(z+2]/\left[1+0.005(z+2)^2\right] \quad (24)$$

$$U(z)=1.8\sin[0.15(z-2)]/\left[1+0.005(z-2)^2\right] \quad (25)$$

and their coefficients of symmetry

$$\Gamma_U(z)=-1 \quad \Gamma_V(z)=-1 \quad (26)$$

with coefficients of symmetry

$$a=2,\ \theta=0,\ \mu=U(0). \quad (27)$$

Errors markers $\varepsilon[\bar{a};\mu;\theta]$, $\varepsilon[a;\bar{\mu};\theta]$, $\varepsilon[a;\mu;\bar{\theta}]$ are *pointing* to the values $\tilde{a}$, $\tilde{\mu}$, $\tilde{\theta}$ closest with the parameters a , μ , θ of correct restoration Figure 8.

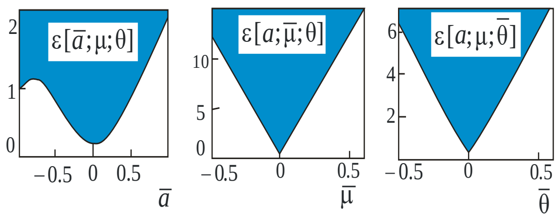
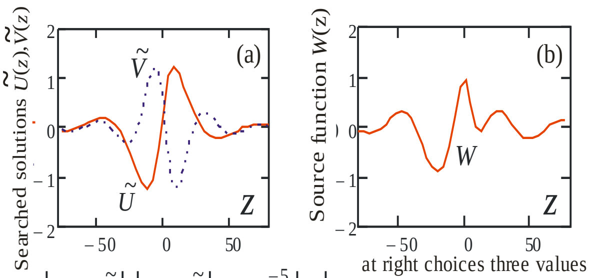


Figure 8

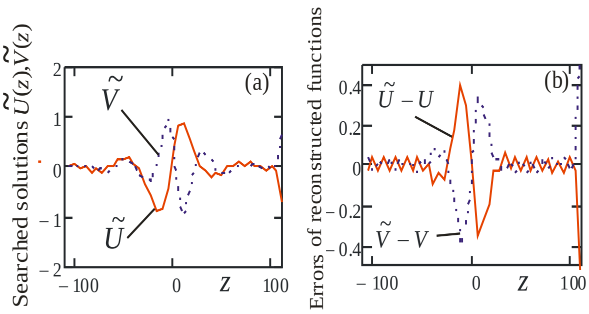
Results $\tilde{U}(z), \tilde{V}(z)$ of restoration of searched functions $U(z), V(z)$ (a) from their tandem $W(z)$ (b), using exact values a, μ, θ , are represented in Fig. 10.



$\left|U-\tilde{U}\right|, \left|V-\tilde{V}\right| < 10^{-5} \left|W\right|_{max}$ $\bar{a}=0,\ \bar{\mu}=0,\ \bar{\theta}=0$

Figure 9

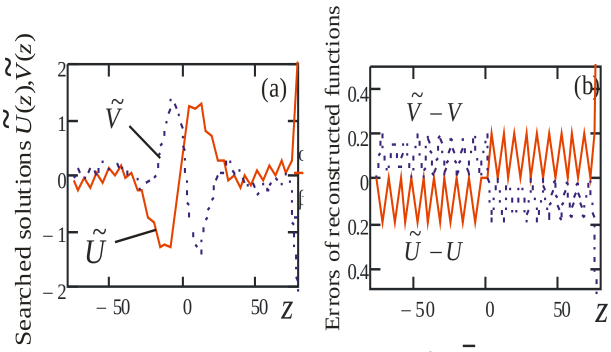
Results $\tilde{U}(z), \tilde{V}(z)$ of restoration of searched functions $U(z), V(z)$ (a) and errors of restoration $\tilde{U}(z)-U(z)$, $\tilde{V}(z)-V(z)$ (b), admitting only one nonzero error $\bar{a} \neq 0$ ($\Delta(U,a) \approx \Delta(V,a) \approx 0.5$, see (15), (16)), are represented in Fig. 10.



error $\bar{a}=0.8$; correct $\bar{\mu}=0$, $\bar{\theta}=0$

Figure 10

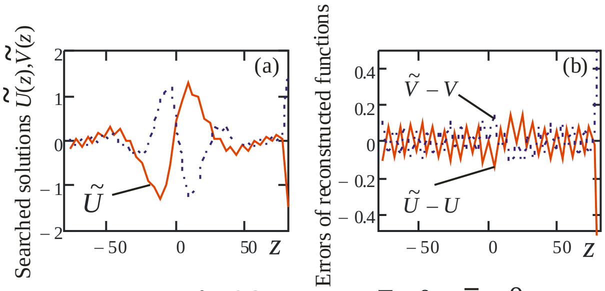
Results $\tilde{U}(z), \tilde{V}(z)$ of restoration of searched functions $U(z), V(z)$ (a) and errors of restoration $\tilde{U}(z)-U(z)$, $\tilde{V}(z)-V(z)$ (b), admitting only one nonzero error $\bar{\mu} \neq 0$, ($\Delta(U,\mu) \approx \Delta(V,\mu) \approx 4.1$, see (17), (18)), are represented in Fig. 11.



error $\bar{\mu}=0.2$;correct $\bar{a}=0$, $\bar{\theta}=0$

Figure 11

Results $\tilde{U}(z), \tilde{V}(z)$ of restoration of searched functions $U(z), V(z)$ (a) and errors of restoration $\tilde{U}(z)-U(z)$, $\tilde{V}(z)-V(z)$ (b), admitting only one nonzero error $\bar{\theta} \neq 0$ ($\Delta(U,\theta) \approx \Delta(V,\theta) \approx 2.0$, see (19), (20),are represented in Fig. 12.



error $\theta = 0.3$; correct $\bar{a} = 0$, $\bar{\mu} = 0$

Figure 12

VIII. Separation of Two Very Closely Spaced Radio-Astronomic Point-Like Sources

For simplicity we consider one-dimensional aperture antenna with characteristic directivity pattern $f(z) = \mathrm{sinc}(35z)$ (all variables are assumed dimensionless). The last means amplitude response of antenna to some source, spaced in antenna's far zone, z is angle coordinate. We spaced two point–like sources in the directions with coordinates $z = -a = -0.005$ and $z = a = 0.005$. Angle distance $2a$ between two sources is less, than the width of directivity pattern main petal. Output of antenna [4] are the powers $U(z) = [\mathrm{sinc}[35(z+a)]]^2$, $V(z) = 0.64[\mathrm{sinc}[35(z-a)]]^2$, created by sources of the antenna's output (amplitude of source in the direction $z = -a = -0.005$ is 1, other source has amplitude 0.8), $\Gamma_V = 1$, $\Gamma_U = 1$, $\theta = 0$. Errors markers (6)-(12) $\varepsilon[\bar{a};\mu;\theta]$, $\varepsilon[a;\bar{\mu};\theta]$, $\varepsilon[a;\mu;\bar{\theta}]$ are pointing to the values $\tilde{a}$, $\tilde{\mu}$, $\tilde{\theta}$ closest with the parameters a , μ , θ of correct restoration Figure 13 and errors $\bar{a} = \tilde{a} - a$, $\bar{\mu} = \tilde{\mu} - \mu$, $\bar{\theta} = \tilde{\theta} - \theta$ are going to zero if we are sorting patterns at . In the case considered the smallness of angle parameter a means antenna's resolution (or superresolution).

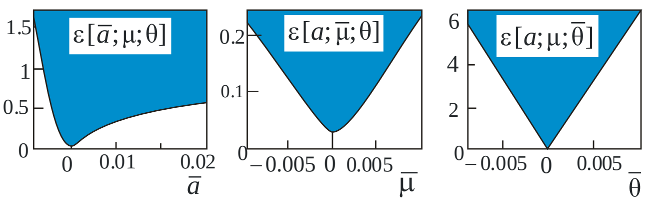
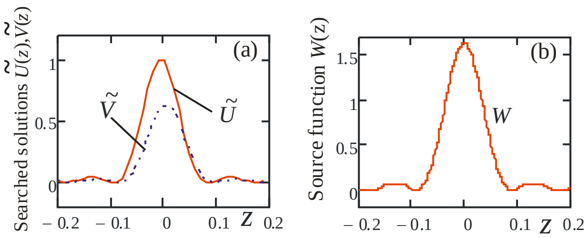


Figure 13

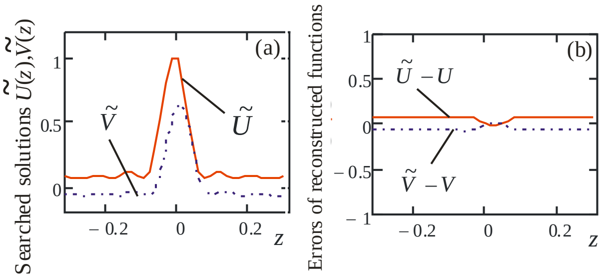
Results $\tilde{U}(z), \tilde{V}(z)$ of restoration of searched (true) functions $U(z), V(z)$ (a) from their tandem $W(z)$ (b), using exact key-numbers a, μ, θ are represented in Figure 14.



$|U - \tilde{U}|, |V - \tilde{V}| < 10^{-5} |W|_{max}$ at right choices three values $\bar{a} = a, \bar{\mu} = 0, \bar{\theta} = 0$

Figure 14

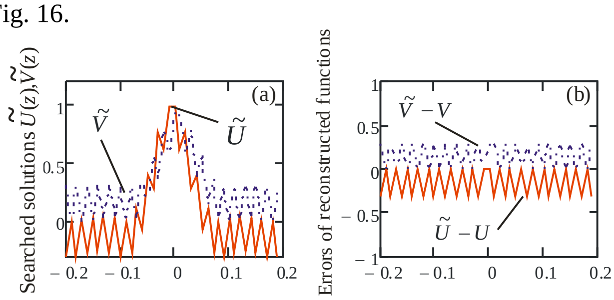
Results $\tilde{U}(z), \tilde{V}(z)$ of restoration of searched functions $U(z), V(z)$ (a) and errors of restoration $\tilde{U}(z) - U(z)$, $\tilde{V}(z) - V(z)$ (b), admitting only one nonzero error $\bar{a} \neq 0$, ($\Delta(U,a) \approx \Delta(V,a) \approx 8.1$ see (15), (16)) are represented in Fig. 15.



error $\bar{a} = 0.001$; correct $\bar{\mu} = 0, \bar{\theta} = 0$

Figure 15

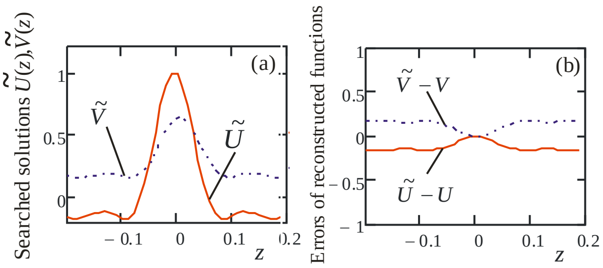
Results $\tilde{U}(z), \tilde{V}(z)$ of restoration of searched functions $U(z), V(z)$ (a) and errors of restoration $\tilde{U}(z) - U(z)$, $\tilde{V}(z) - V(z)$ (b), admitting only one nonzero error $\bar{\mu} \neq 0$, ($\Delta(U,\mu) \approx \Delta(V,\mu) \approx 0.6$ see (17), (18)) are represented in Fig. 16.



error $\bar{\mu} = 0.3$; correct $\bar{a} = 0, \bar{\theta} = 0$

Figure 16

Results $\tilde{U}(z), \tilde{V}(z)$ of restoration of searched functions $U(z), V(z)$ (a) and errors of restoration $\tilde{U}(z) - U(z)$, $\tilde{V}(z) - V(z)$ (b), admitting only one nonzero error $\bar{\theta} \neq 0$, ($\Delta(U,\theta) \approx \Delta(V,\theta) \approx 0.25$, see (19), (20)) are represented in Fig. 17.



error $\bar{\theta} = 0.2$; correct $\bar{a} = 0$, $\bar{\mu} = 0$

Figure 17

IX. Two Gaussian Functions

In this section we will test the restoration procedure (section IV) on the case of two Gaussian functions

$$U(z) = 2.2\exp[-0.005(z+2)^2] \qquad (32)$$

$$V(z) = 1.8\exp[-0.005(z-2)^2] \qquad (33)$$

with coefficients of symmetry

$$\Gamma_U(z) = 1, \ \Gamma_V(z) = 1. \qquad (34)$$

And true combination of key-numbers

$$a = 2\,,\ \theta = 0\,,\ \mu = U(0) \qquad . \qquad (35)$$

Errors markers $\varepsilon[\bar{a};\mu;\theta]$, $\varepsilon[a;\bar{\mu};\theta]$, $\varepsilon[a;\mu;\bar{\theta}]$ are *pointing* to the values $\tilde{a}$, $\tilde{\mu}$, $\tilde{\theta}$ closest with the parameters a, μ, θ of correct restoration are represented in Figure 18

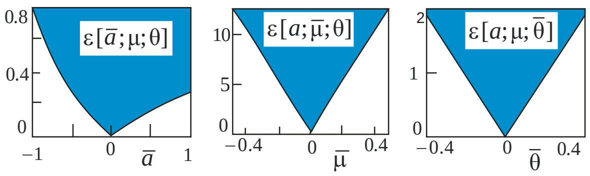


Figure 18

Results $\tilde{U}(z),\tilde{V}(z)$ of restoration of searched functions $U(z),V(z)$ (a) from their tandem $W(z)$ (b), using exact values a,μ,θ, are represented in Fig. 19.

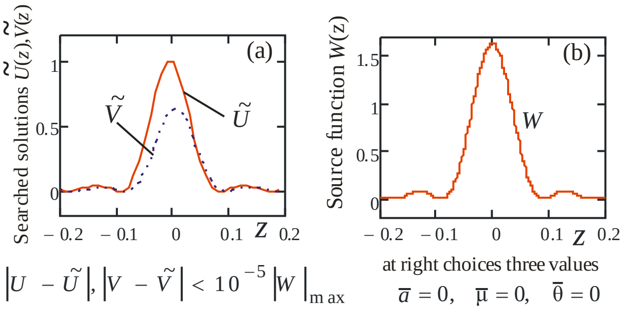


Figure 19

Results $\tilde{U}(z),\tilde{V}(z)$ of restoration of searched functions $U(z),V(z)$ (a) and errors of restoration $\tilde{U}(z)-U(z)$, $\tilde{V}(z)-V(z)$ (b), admitting only one nonzero error $\bar{a}\neq 0$, ($\Delta(U,a)\approx\Delta(V,a)\approx 19$, see (15), (16)) are represented in Fig. 20.

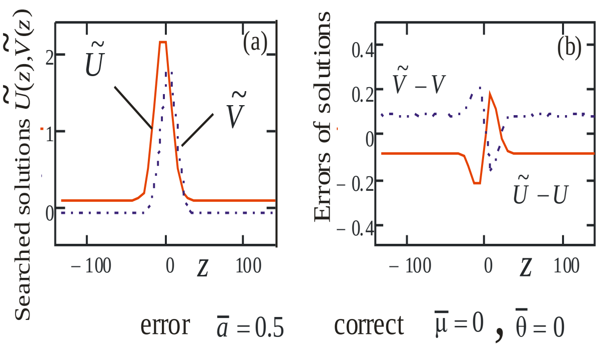


Figure 20

Results $\tilde{U}(z),\tilde{V}(z)$ of restoration of searched functions $U(z),V(z)$ (a) and errors of restoration $\tilde{U}(z)-U(z)$, $\tilde{V}(z)-V(z)$ (b), admitting only one nonzero error $\bar{\mu}\neq 0$, ($\Delta(V,\mu)\approx\Delta(U,\mu)\approx 1.2$, see (17), (18)) are represented in Figure 21.

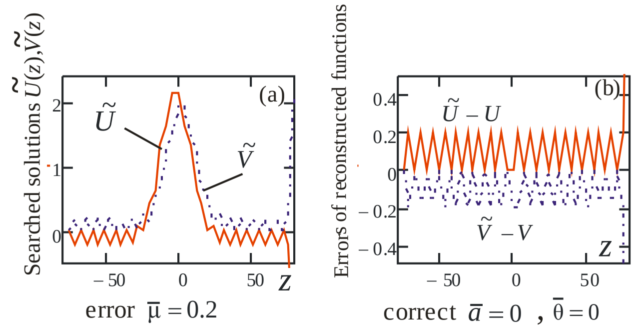


Figure 21

Results $\tilde{U}(z),\tilde{V}(z)$ of restoration of searched functions $U(z),V(z)$ (a) and errors of restoration $\tilde{U}(z)-U(z)$, $\tilde{V}(z)-V(z)$ (b), admitting only one nonzero error $\bar{\theta}\neq 0$ ($\Delta(V,\theta)\approx\Delta(U,\theta)\approx 1.1$ (see (19), (20)), are represented in Figure 22.

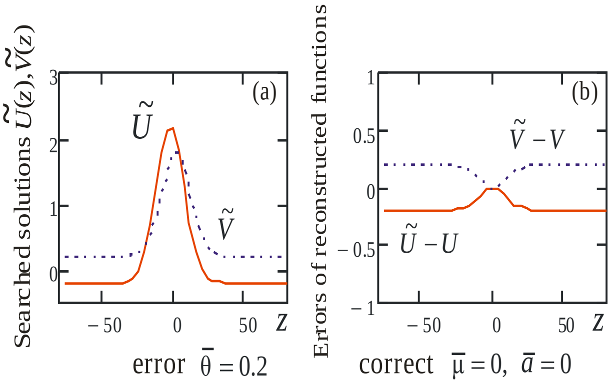


Figure 22

X. Identical and Different Searched Functions. Asymmetric Source Function.

Figure 23-a represents the different error markers $\varepsilon = \varepsilon(a;\mu;\theta)$, corresponding to searched Lorentz's functions $U(z) = 1.2[1+0.005(z+2)^2]$, $V(z) = 2.2[1+0.005(z-2)^2]$ with different amplitudes (1.2 and 2.2 continuous line) and $U(z) = 2.2[1+0.005(z+2)^2]$, $V(z) = 2.2[1+0.005(z-2)^2]$ ($\Gamma_U = \Gamma_V = 1$) with identical amplitudes (2.2 and 2.2, dotted line). We see, that case of different amplitudes (and asymmetric $W(z)$) gives us more sharp shape of error markers $\varepsilon[\tilde{a};\tilde{\mu};\tilde{\theta}]$ and more easy finding of key-numbers a, μ, θ, than the case of identical amplitudes (and symmetric $W(z)$). More great difference (Figure 23-b) in the behavior of error markers $\varepsilon[\tilde{a};\tilde{\mu};\tilde{\theta}]$ we see in the cases of Gaussian functions with different amplitudes (1.2 and 2.2, continuous line) $U(z) = 1.2\exp[-((z+2)/6)^2]$, $V(z) = 2.2\exp[-((z-2)/6)^2]$ and the pair of Gaussian functions with identical amplitudes $U(z) = 2.2\exp[-((z+2)/6)^2]$, $V(z) = 2.2\exp[-((z-2)/6)^2]$ (2.2 and 2.2, dotted line). It is easy to see, that error marker $\varepsilon[\tilde{a};\tilde{\mu};\tilde{\theta}]$ is quite non-sensitive on interval $0 < a < 2$.

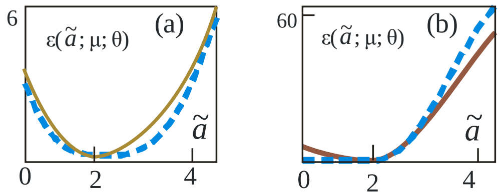


Figure 23

Now we consider symmetric source function $W(z)$ as only one Gaussian function

$$W(z) = W_G(z) = 4\exp\left[-(z/40)^2\right] \qquad (36)$$

(Figure 24-a) or only one Lorentz's function

$$W(z) = W_L(z) = 4/\left[1+(z/40)^2\right] \qquad (37)$$

(Figure 24-b). Let's initiate the search (above section IV) for tandem of symmetric ($\Gamma_U = \Gamma_V = 1$) functions on the same carrier $|z| \le L$, observing behavior of the marker of the error $\varepsilon[\tilde{a};\tilde{\mu};\tilde{\theta}]$. In this case amplitudes of pairs of searched symmetric functions can be of *only identical amplitudes*. In the case of Gaussian source function $W_G(z)$ the calculation procedure of section IV generates single pair "U,V" of smooth symmetric "own functions", attached to concrete single "own number" of the "continuum spectrum of own numbers"

$$3 < a < 10\,. \qquad (38)$$

On the other hand, Lorentz's source function generates only one pair "U,V", but error marker $\varepsilon[\tilde{a};\tilde{\mu};\tilde{\theta}]$ does not show the key-number a sufficiently clear.

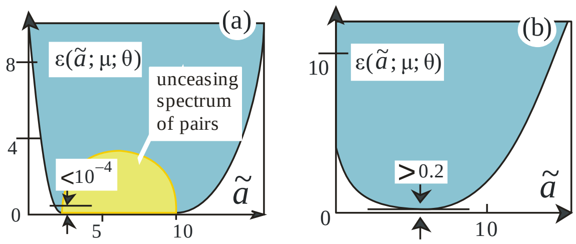


Figure 24

Graph of marker $\varepsilon[\tilde{a};\mu;\theta]$ in Figure 24 means, for instance, that calculation procedure (described in Section IV) gives true solution $U(z)$ of the equation

$$U(z-a) + U(z+a) = W_G(z) \qquad (39)$$

for each value a from interval (38). Taking into account the condition

$$U(z-a) = U(z+a)\,, \qquad (41)$$

(guaranteed by Section IV), we can see, that

$$2U(z-\alpha) = W_G(z) \quad \text{or} \quad 2U(z+\alpha) = W_G(z)\,. \qquad (42)$$

In other words, Gaussian function $W_G(z)$ can be represented by the pair (39) of Gaussian functions $U(z-a)$ and $U(z+a)$ with shift a in the interval (38).

XI. Conclusion

Main conclusion: to separate both similar continuous smooth symmetric function of given their tandem (doing it by single way) we *need only three true key-numbers*, known before or founded by sorting patterns. Represented numerical results have an empirical nature and seemed sufficiently unexpected. This is why we used many graphs in this paper.

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(b) design of floating devises, which convert boat oscillations into a periodical sequence of one polarity stream pulses ("wave engine").

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